

# SOME GENERAL PROPERTIES OF LAD AND RAD AG-GROUPOIDS

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**ABSTRACT.** A groupoid that satisfies the left invertive law:  $ab \cdot c = cb \cdot a$  is called an AG-groupoid. We extend the concept of left abelian distributive groupoid (LAD) and right abelian distributive groupoid (RAD) to introduce new subclasses of AG-groupoid, left abelian distributive AG-groupoid and right abelian distributive AG-groupoid. We give their enumeration up to order 6 and find some basic relations of these new classes with other known subclasses of AG-groupoids and other relevant algebraic structures. We establish a method to test an arbitrary AG-groupoid for these classes.

## 1. Introduction

An AG-groupoid is a generalization of commutative semigroup, in which the left invertive law:  $(ab)c = (cb)a$  holds [1]. A groupoid  $S$  is called left (resp. right) abelian distributive groupoids if it satisfies  $a \cdot bc = ab \cdot ca$  (resp.  $ab \cdot c = ca \cdot bc$ ) [2]. Here we extend these concepts to left (right) abelian distributive AG-groupoids. We prove their existence by providing non-associative examples of various finite orders. We also prove their relations with some of the already known subclasses [4, 6, 8, 11, 13, 15] of AG-groupoid and with other algebraic structures. AG-groupoids are enumerated up to order 6 in [4] using GAP [3] by one of our author. We also use the same technique to enumerate our new subclasses of AG-groupoids. Table 1, contains the counting of these new subclasses of AG-groupoids. An AG-groupoid  $S$  is called monoid if it contains a unique left identity. Every monoid satisfies the paramedial

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property. It is also interesting to note that if  $S$  possesses the right identity element then it becomes a semigroup.  $S$  is called medial if it satisfies the identity,  $ab.cd = ac.bd$ . It is easy to prove that every AG-groupoid is medial. AG-groupoids have a variety of applications in flocks and geometry[1, 7]. In the following we give some definitions of AG-groupoids with their identities that will be used in the rest of this article.

An Abel-Grassmann groupoid  $S$  [1], abbreviated as an AG-groupoid, is a groupoid that satisfies the left invertive law,

$$(1.1) \quad ab \cdot c = cb \cdot a \quad \forall a, b, c \in S$$

An AG-groupoid  $S$  is called —

- (i) — right commutative AG-groupoid, if  $a \cdot bc = a \cdot cb$  [6].
- (ii) — self-dual AG-groupoid, if  $a \cdot bc = c \cdot ba$  [7].
- (iii) — left distributive (LD) AG-groupoid, if  $a \cdot bc = ab \cdot ac$  [7].
- (iv) — right distributive (RD) AG-groupoid, if  $ab \cdot c = ac \cdot bc$  [7].
- (v) — an AG\*\*-groupoid if it satisfies the identity  $a(bc) = b(ac)$  [7].
- (vi) — flexible AG-groupoid if it satisfies the identity  $a(ba) = (ab)a$  [7].
- (vii) — paramedial AG-groupoid if it satisfies the identity  $ab.cd = db.ca$  [14].
- (viii) — medial AG-groupoid if it satisfies the identity  $ab.cd = ab.cd$  [14].
- (ix) — right commute AG-groupoid if it satisfies the identity  $a \cdot bc = a \cdot cb$ .

Order	3	4	5	6
Total (AG-groupoids)	20	331	31913	40104513
Non associative RAD AG-groupoids	6	175	21186	34539858
Non associative LAD AG-groupoids	0	0	27	1106
Non associative AD AG-groupoids	0	0	0	0

TABLE 1. Enumeration of RAD & LAD AG-groupoids up to order 6.

## 2. Left Abelian Distributive AG-groupoids

**Definition 1.** An AG-groupoid  $S$  is called left abelian distributive AG-groupoid, denoted by LAD if  $\forall a, b, c \in S$ ,

$$(2.1) \quad a \cdot bc = ab \cdot ca$$

**Example 1.** Let  $S = \{1, 2, 3, 4\}$  with the following table. Then it is easy to verify that  $S$  is LAD AG-groupoid.

*	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	1	3

## 3. Left Abelian Distributive AG-test

In this section we discuss a procedure of [12] that whether an AG-groupoid  $(G, \cdot)$  is LAD-AG-groupoid or not, for this we define the following binary operation

$$(3.1) \quad a \odot b = ab \cdot xa$$

$$(3.2) \quad a \circ b = a \cdot bx$$

The law  $a \cdot bx = ab \cdot xa$  is satisfied if,

$$(3.3) \quad a \odot b = a \circ b$$

To construct the table of operation “ $\odot$ ” for any fixed  $x \in G$ . We rewrite  $x$ -row of the “ $\cdot$ ” table as an index row for the new extended table. Multiply corresponding elements of each row of the original table by the corresponding element of the each index row for the extended table. Similarly the table of the operation “ $\circ$ ” for any fixed  $x \in G$  is obtained by rewriting the  $x$ -column of “ $\cdot$ ” table as an index column and multiplying it by the elements of the index row from the left. If the tables obtained for the operation “ $\odot$ ” and “ $\circ$ ” coincides for all  $x \in G$ , then the equation 2.1 holds and the AG-groupoid is an LAD-AG-groupoid in that case. We illustrate the procedure with the following example.

**Example 2.** *Verify the AG-groupoid  $G$  with the following table for LAD- AG-groupoid.*

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	2	2	2
4	1	2	2	2	3
5	1	2	2	2	4

Extend the above table in the way as described above we get the following table, where the tables on right to the original table is for the operation “ $\odot$ ” and the tables that lies downwards of the original tables are constructed for the operation “ $\circ$ ”.

It is clear from the extended table that the downward tables and the tables on the right coincide so  $G$  is LAD-AG-groupoid.

#### 4. CHARACTERIZATION OF LAD-AG-GROUPOID

**Theorem 1.** *Let  $S$  be an LAD-AG-groupoid. then the following hold.*

*$S$  is RC-AG-groupoid*

- (i)  *$S$  is right commutative (RC) AG-groupoid;*
- (ii)  *$S$  is self-dual AG-groupoid;*
- (iii)  *$S$  is  $AG^{**}$ -groupoid;*
- (iv)  *$S$  is left distributive AG-groupoid.*

*Proof.* Let  $S$  be an LAD-AG-groupoid and  $a, b, c \in S$ . Then

- (i) Using the Identity 2.1 and the medial law, we have  $a \cdot bc = ab \cdot ca = ac \cdot ba = a \cdot cb$ . Thus  $S$  is RC-AG-groupoid.

[illegible]

TABLE 2

get,

$$\begin{aligned} a \cdot bc &= ab \cdot ca = (ab \cdot c)(a \cdot ab) = (ab \cdot c)(a \cdot ba) = (ab \cdot a)(c \cdot ba) = \\ &= (ab \cdot a)(cb \cdot ac) = (ab \cdot cb)(a \cdot ac) = (ab \cdot cb)(aa \cdot ca) = \\ &= (ab \cdot aa)(cb \cdot ca) = (ab \cdot aa)(cb \cdot ac) = (a \cdot ba)(c \cdot ba) = \\ &= (ac)(ba \cdot ba) = (ac)(ba \cdot ab) = (ac)(b \cdot aa) = (ac)(aa \cdot b) = \\ &= (ac)(ba \cdot a) = c \cdot ba \Rightarrow a \cdot bc = c \cdot ba. \end{aligned}$$

Hence  $S$  is self-dual AG-groupoid.

(iii) Using the Identity 2.1, medial law and Theorem (1 (i)), we get

$$a \cdot bc = a \cdot cb = b \cdot ca = bc \cdot ab = ba \cdot cb = b \cdot ac.$$

Hence  $S$  is AG<sup>\*\*</sup>-groupoid.

(iv) The Identity 2.1 and medial law imply that

$$\begin{aligned} a \cdot bc &= ab \cdot ca = (ab \cdot c)(a \cdot ab) \\ &= (ab \cdot a)(c \cdot ab) = ab \cdot ac \\ \Rightarrow a \cdot bc &= ab \cdot ac. \end{aligned}$$

Hence  $S$  is left distributive AG-groupoid. ■

**Corollary 1.** *Every LAD-AG-groupoid is paramedial AG-groupoid and hence is a left nuclear square AG-groupoid.*

It is worth mentioning that the concepts of LD and LAD for AG-groupoids are different. To this end we give an example of LD-AG-groupoid which is not LAD-AG-groupoid.

**Example 3.** *Let  $S = \{1, 2, 3, 4\}$  with the following table. Then one can easily verify that  $S$  is an LD-AG-groupoid. Since  $1 \cdot 24 \neq 12 \cdot 41$  thus  $S$  is not an LAD-AG-groupoid.*

*	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

## 5. RAD-AG-groupoid

**Definition 2.** *An AG-groupoid  $S$  is called right abelian distributive (or shortly RAD) AG-groupoid if  $\forall a, b, c \in S$ ,*

$$(5.1) \quad ab \cdot c = ca \cdot bc$$

**Example 4.** *Let  $S = \{a, b, c\}$  with the following table then one can easily verify that  $S$  is an RAD AG-groupoid.*

$*$	$a$	$b$	$c$
$a$	$a$	$a$	$a$
$b$	$a$	$a$	$a$
$c$	$a$	$b$	$a$

## 6. RAD-AG-Test

In this section we discuss the procedure that how to check an arbitrary AG-groupoid  $(G, \cdot)$  for an RAD-AG-groupoid, for this we define the following binary operation

$$(6.1) \quad a \circ b = xa \cdot bx$$

$$(6.2) \quad a \diamond b = ab \cdot x$$

The law  $ab \cdot x = xa \cdot bx$  is satisfied if;

$$(6.3) \quad a \circ b = a \diamond b$$

To construct table of operation “ $\circ$ ” for any fixed  $x \in G$ , rewriting  $x$ -row of the “ $\cdot$ ” table as an index row of the new extended table and multiply its elements by the elements of the  $x$ -column of “ $\cdot$ ” table turn by turn to list the rows of “ $\circ$ ” tables.

Similarly the tables of the operation “ $\diamond$ ” for any fixed  $x \in G$  is constructed by multiplying a fixed element  $x \in G$  by elements of the “ $\cdot$ ” table from the left. If the tables obtained for the operation “ $\circ$ ” and “ $\diamond$ ” coincide for all  $x \in G$ , then 5.1 holds, and the AG-groupoid is an RAD-AG-groupoid.

**Example 5.** Check the following AG-groupoid  $G = \{1, 2, 3\}$  for RAD-AG-groupoid.

$\cdot$	1	2	3
1	1	1	1
2	1	1	1
3	2	2	2

Extend the above table in the way as described above we get the following:

$\cdot$	1	2	3	1	1	1	1	1	1	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1
3	2	2	2	1	1	1	1	1	1	1	1	1
1	1	1	1									
	1	1	1									
	1	1	1									
2	1	1	1									
	1	1	1									
	1	1	1									
3	1	1	1									
	1	1	1									
	1	1	1									

It is clear from the extended table that the downward tables for the operation “ $\diamond$ ” and the tables on the right for the operation “ $\circ$ ” coincide so  $G$  is an RAD-AG-groupoid.

**Theorem 2.** *Every RAD-AG-groupoid is right distributive AG-groupoid.*

*Proof.* Let  $S$  be an RAD-AG-groupoid, and let  $a, b, c \in S$ . Then by the Identity 5.1 and medial law, we get

$$\begin{aligned}
 ab \cdot c &= ca \cdot bc = (bc \cdot c)(a \cdot bc) \\
 &= (bc \cdot a)(c \cdot bc) = ac \cdot bc \\
 \Rightarrow ab \cdot c &= ac \cdot bc.
 \end{aligned}$$

Hence  $S$  is right distributive AG-groupoid. ■

The converse of the previous theorem is not valid. The following example show that not every RD-AG-groupoid is an RAD-AG-groupoid.

**Example 6.** *Let  $S = \{a, b, c, d\}$  with the following operation. Then one can easily verify that  $S$  is an RD-AG-groupoid, but is not an RAD-AG-groupoid.*

$*$	$a$	$b$	$c$	$d$
$a$	$a$	$c$	$d$	$b$
$b$	$d$	$b$	$a$	$c$
$c$	$b$	$d$	$c$	$a$
$d$	$c$	$a$	$b$	$d$



## 7. Abelian Distributive AG-groupoid

**Definition 3.** *An AG-groupoid  $S$  is called abelian distributive AG-groupoid denoted by AD-AG-groupoid if it is both RAD and LAD AG-groupoids.*

The following result confirms that non-associative AD-AG-groupoids do not exist.

**Theorem 3.** *Every AD-AG-groupoid is a semigroup.*

*Proof.* Let  $S$  be an AD-AG-groupoid, and let  $a, b, c \in S$ . Then by identities (2.1, 5.1, 1.1), and medial law we get,

$$a \cdot bc = ab \cdot ca = ac \cdot ba = cb \cdot a = ab \cdot c \Rightarrow a \cdot bc = ab \cdot c.$$

Hence  $S$  is a semigroup. ■

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